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A Microinstability Code for a Uniform Magnetized Plasma with an Arbitrary Distribution Function

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We have developed a very general computer code for studying microinstabilities in a uniform magnetized plasma. Employing a new algorithm to perform two-dimensional numerical integrals in the conductivity tensor, the code can handle an arbitrary distribution function given by either an analytical function or numerical values on a momentum space grid and solve the full dispersion relation for an arbitrary propagation angle in either a non-relativistic or relativistic plasma except for a highly relativistic plasma (energy $\gg 1$ MeV). The results for cyclotron–maser instability and whistler–wave instability are presented to illustrate the validity of the method.

1. INTRODUCTION

Linear microstability of a uniform magnetized plasma has been extensively studied both analytically and numerically [1]. For an analytical model distribution such as Maxwellian, bi-Maxwellian, and Dory–Guest–Harris (DGH) distribution [2], one can reduce the conductivity tensor (and dispersion relation) to an analytical expression consisting of the modified Bessel functions and the plasma dispersion function [3] in non-relativistic cases. In relativistic cases, however, it is generally not possible to analytically reduce the conductivity tensor to a combination of these and/or other known functions, and therefore it is necessary to perform numerical integrals in the momentum space, or make approximations to render an analytical evaluation tractable. Results then have a limited range of validity with respect to wavenumber, energy, type of distribution, and so forth.

Even in a non-relativistic case, one often encounters a situation where it is very difficult to represent a velocity distribution function by a combination of known functions. Such a situation occurred, for example, for a tandem-mirror plasma with electron cyclotron resonance heating (ECRH) [4]. Fokker–Planck studies of electrons heated by ECRH in a magnetic mirror showed an electron distribution with extended tail and a loss cone which may only be approximated by a combination of many functions [5]. This usually is done by a trial and error method and often results in a poor fit with part of the velocity space having negative values [6]. Not only in mirror machines but also in tokamaks and other devices there exist non-Maxwellian plasmas due to auxiliary heating such as ECRH, ion-cyclotron heating, and neutral-beam heating. In addition, in plasma heating by use of a free-electron-laser on the MTX device at LLNL a highly non-Maxwellian distribution with a loss cone is expected to be generated because of a high-power, short-pulse electric field of several hundred kV/cm [7].

There have been only a few computer codes developed so far for studying the dispersion characteristics, absorption, and emission in a relativistic non-Maxwellian plasma. (There are numerous works published on the propagation and absorption in a relativistic Maxwellian plasma, however. See, for example, Ref. [8].) Weiss [9] presented an elegant and general formulation, that is applicable to both non-relativistic and relativistic plasmas with an arbitrary distribution, and reported results for an isotropic distribution. Following Weiss' method, Tamor [10] developed a code for an arbitrary distribution function in the energy range of 50–500 keV. He presented the results for a loss-cone distribution and a bi-Maxwellian distribution. Yoon and Chang [11] derived an exact dielectric tensor for a relativistic distribution with loss cone and parallel drift and considered a weakly relativistic limit.

Neither of the works mentioned above, however, treats a truly arbitrary distribution function in a sense that they still have to employ a model distribution represented by a combination of Maxwellian, bi-Maxwellian, DGH distribution, and/or a known analytical function. As described earlier it is often difficult to fit an actual distribution observed in an experiment or obtained from another computer code. In addition they restricted themselves to real frequencies, while we retain complex frequencies so that we can treat absolute instability. One can, of course, employ Weiss' method for our purpose, but then one has to deal with Bessel functions of complex order.

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The purpose of our article is twofold. We first present a general method of evaluating the conductivity tensor for either non-relativistic or relativistic, arbitrary distributions. Second, we show how one can handle numerical distribution functions defined on a velocity or momentum grid. The method, of course, works well with distributions given by any analytical functions.

The general dispersion relation for linear waves in a uniform, infinite plasma in a uniform magnetic field, $\mathbf{B}_0 = B_0\hat{z}$, is [12]

$$\left|\left(1-\frac{c^2k^2}{\omega^2}\right)\mathbf{I}+\frac{c^2}{\omega^2}\mathbf{kk}+\frac{4\pi i}{\omega}\boldsymbol{\sigma}\right|=0, \tag{1}$$

where $\mathbf{k}=\mathrm{k}_\perp\hat{x}+k_{||}\hat{z}$, ω is the complex frequency, c is the speed of light, $\mathbf{I}$ is the unit dyadic, and $\boldsymbol{\sigma}$ is the conductivity tensor. The conductivity tensor is given by

$$\begin{aligned}\boldsymbol{\sigma}=\sum_j\frac{2\pi e_j^2}{m_j}\Bigg\{&\iint dv_\perp\,dv_{||}\left[\frac{\partial F_j}{\partial v_\perp}-\frac{k_{||}}{\gamma\omega}\right.\\&\left.\times\left(v_{||}\frac{\partial F_j}{\partial v_\perp}-v_\perp\frac{\partial F_j}{\partial v_{||}}\right)\right]\sum_l\frac{\mathbf{M}^l}{\gamma\omega-l\Omega_j-k_{||}v_{||}}\\&+\hat{z}\hat{z}\iint dv_\perp\,dv_{||}\frac{v_{||}}{\gamma\omega}\left(v_\perp\frac{\partial F_j}{\partial v_{||}}-v_{||}\frac{\partial F_j}{\partial v_\perp}\right)\Bigg\},\end{aligned} \tag{2}$$

where standard notation is used and

$$\mathbf{M}^l=\begin{bmatrix}-i\dfrac{\Omega_j^2}{k_\perp^2}l^2J_l^2(b_j) & s_j\dfrac{\Omega_j}{k_\perp}lJ_l(b_j)J_l'(b_j)v_\perp & -i\dfrac{\Omega_j}{k_\perp}lJ_l^2(b_j)v_{||}\\ -M_{xy}^l & -iJ_l'^2(b_j)v_\perp^2 & -s_jJ_l(b_j)J_l'(b_j)v_\perp v_{||}\\ M_{xz}^l & -M_{yz}^l & -iJ_l^2(b_j)v_{||}^2\end{bmatrix}.$$

Here, s_j represents the sign of the charge of species j, and $b_j=k_\perp v_\perp/\Omega_j$.

For a bi-Maxwellian distribution of particle species j given by

$$F_j=n_j\left(\frac{m_j}{2\pi T_{\perp j}}\right)^{3/2}\left(\frac{T_{\perp j}}{T_{||j}}\right)^{1/2}\times\exp\left[-\frac{m_j}{2}\left(\frac{v_\perp^2}{T_{\perp j}}+\frac{v_{||}^2}{T_{||j}}\right)\right], \tag{3}$$

one obtains in a non-relativistic case,

$$\begin{aligned}\sigma_{xx}=&\,i\frac{n_je_j^2}{m_j}\frac{1}{k_{||}\alpha_{||j}}\frac{e^{-\lambda}}{\lambda}\sum_l l^2I_l(\lambda)\\&\times\left[\left(1-\frac{T_{\perp j}}{T_{||j}}\right)\frac{k_{||}\alpha_{||j}}{\omega}-\left(\frac{l\Omega_j}{\omega}+\frac{T_{\perp j}}{T_{||j}}\frac{\omega-l\Omega_j}{\omega}\right)Z(\xi_j)\right]\end{aligned}$$

$$\begin{aligned}\sigma_{xy}=&\frac{n_je_j^2s_j}{m_j}\frac{e^{-\lambda}}{k_{||}\alpha_{||j}}\sum_l l[I_l(\lambda)-I_l'(\lambda)]\\&\times\left[\left(1-\frac{T_{\perp j}}{T_{||j}}\right)\frac{k_{||}\alpha_{||j}}{\omega}-\left(\frac{l\Omega_j}{\omega}+\frac{T_{\perp j}}{T_{||j}}\frac{\omega-l\Omega_j}{\omega}\right)Z(\xi_j)\right]\end{aligned}$$

$$\begin{aligned}\sigma_{xz}=&-i\frac{n_je_j^2}{m_j}\frac{k_\perp}{k_{||}}\frac{e^{-\lambda}}{\lambda}\sum_l l\,I_l(\lambda)\\&\times\left[\left(1-\frac{T_{\perp j}}{T_{||j}}\right)\frac{l\Omega_j}{\omega}\right.\\&\left.+\left(\frac{l\Omega_j}{\omega}+\frac{T_{\perp j}}{T_{||j}}\frac{\omega-l\Omega_j}{\omega}\right)\frac{\omega-l\Omega_j}{k_{||}\alpha_{||j}}Z(\xi_j)\right]\end{aligned}$$

$$\sigma_{yx}=-\sigma_{xy}$$

$$\begin{aligned}\sigma_{yy}=&\,i\frac{n_je_j^2}{m_j}\frac{e^{-\lambda}}{k_{||}\alpha_{||j}}\\&\times\sum_l\left[\left(\frac{l^2}{\lambda}+2\lambda\right)I_l(\lambda)-2\lambda I_l'(\lambda)\right]\\&\times\left[\left(1-\frac{T_{\perp j}}{T_{||j}}\right)\frac{k_{||}\alpha_{||j}}{\omega}-\left(\frac{l\Omega_j}{\omega}+\frac{T_{\perp j}}{T_{||j}}\frac{\omega-l\Omega_j}{\omega}\right)Z(\xi_j)\right]\end{aligned}$$

$$\begin{aligned}\sigma_{yz}=&\frac{n_je_j^2s_j}{m_j}\frac{k_\perp}{\Omega_jk_{||}}e^{-\lambda}\sum_l[I_l(\lambda)-I_l'(\lambda)]\\&\times\left[\left(1-\frac{T_{\perp j}}{T_{||j}}\right)\frac{l\Omega_j}{\omega}\right.\\&\left.+\left(\frac{l\Omega_j}{\omega}+\frac{T_{\perp j}}{T_{||j}}\frac{\omega-l\Omega_j}{\omega}\right)\frac{\omega-l\Omega_j}{k_{||}\alpha_{||j}}Z(\xi_j)\right]\end{aligned}$$

$$\sigma_{zx}=\sigma_{xz}$$

$$\sigma_{zy}=-\sigma_{yz}$$

$$\begin{aligned}\sigma_{zz}=&-i\frac{2n_je_j^2}{m_j}\frac{T_{||j}}{T_{\perp j}}\frac{e^{-\lambda}}{k_{||}\alpha_{||j}}\sum_l I_l(\lambda)\\&\times\left(\frac{l\Omega_j}{\omega}+\frac{T_{\perp j}}{T_{||j}}\frac{\omega-l\Omega_j}{\omega}\right)\frac{\omega-l\Omega_j}{k_{||}\alpha_{||j}}[1+\xi_jZ(\xi_j)],\end{aligned} \tag{4}$$

where $I_l(\lambda)$ is the modified Bessel function, Z is the plasma dispersion function, $\alpha_{\perp j}^2=2T_{\perp j}/m_j$, $\alpha_{||j}^2=2T_{||j}/m_j$, $\lambda=k_\perp^2\alpha_{\perp j}^2/2\Omega_j^2$, and $\xi_j=(\omega-l\Omega_j)/k_{||}\alpha_{||j}$.

2. NUMERICAL PROCEDURE

To simplify notation for the purpose of illustrating our procedure, we consider only a single species, e.g., electrons, in evaluating the conductivity tensor and employ the following dimensionless units: $u_{||}=v_{||}/c$, $u_\perp=v_\perp/c$, $W=\omega/\Omega$, $K_{||}=ck_{||}/\Omega$, $K_\perp=ck_\perp/\Omega$, where Ω is the

cyclotron frequency. Note that $\gamma = (1 + u_\perp^2 + u_\parallel^2)^{1/2}$. The conductivity tensor, $\boldsymbol{\sigma}$, then becomes

$$\boldsymbol{\sigma} \propto \iint du_\perp\, du_\parallel \left[\frac{\partial F}{\partial u_\perp} - \frac{K_\parallel}{\gamma W}\left(u_\parallel \frac{\partial F}{\partial u_\perp} - u_\perp \frac{\partial F}{\partial u_\parallel}\right)\right] \times \sum_l \frac{\mathbf{M}}{\gamma W - l - K_\parallel u_\parallel} + \hat{z}\hat{z} \iint du_\perp\, du_\parallel \frac{u_\parallel}{\gamma W}\left(u_\perp \frac{\partial F}{\partial u_\parallel} - u_\parallel \frac{\partial F}{\partial u_\perp}\right), \tag{5}$$

where

$$\mathbf{M} = \begin{bmatrix} -i\dfrac{l^2 J_l^2}{K_\perp^2} & (su_\perp/K_\perp)\, l J_l J_l' & -(iu_\parallel/K_\perp)\, l J_l^2 \\ -M_{xy} & -iu_\perp^2 J_l'^2 & -su_\perp u_\parallel J_l J_l' \\ M_{xz} & -M_{yz} & -iu_\parallel^2 J_l^2 \end{bmatrix}.$$

The second integral in Eq. (5) is trivial. For the first integral, the resonance term can be written as

$$\frac{1}{\gamma W - l - K_\parallel u_\parallel} = \frac{\gamma W + l + K_\parallel u_\parallel}{(W^2 - K_\parallel^2)(u^+ - u^-)} \times \left(\frac{1}{u_\parallel - u^+} - \frac{1}{u_\parallel - u^-}\right), \tag{6}$$

where $u^\pm$ are the resonant velocities, i.e., the roots of the quadratic equation,

$$(\gamma W)^2 - (l + K_\parallel u_\parallel)^2 = 0, \tag{7}$$

and given by

$$u^\pm = \frac{1}{W^2 - K_\parallel^2}\{lK_\parallel \pm W \times [l^2 + (K_\parallel^2 - W^2)(1 + u_\perp^2)]^{1/2}\}. \tag{8}$$

The integrals needed to evaluate all the elements of the conductivity tensor, Eq. (2), can now be cast into the form, except for one term in the zz-element,

$$S = \frac{1}{W^2 - K_\parallel^2}\int_0^\infty du_\perp \frac{\mathscr{Z}(u_\perp)}{u^+ - u^-} \times \left\{\int_{-\infty}^{\infty} du_\parallel \left(\frac{1}{u_\parallel - u^+} - \frac{1}{u_\parallel - u^-}\right) \times H(u_\parallel, u_\perp) + i\pi[\sigma_a^+ H(u^+, u_\perp) - \sigma_a^- H(u^-, u_\perp)]\right\}, \tag{9}$$

where $\mathscr{Z}(u_\perp)$ involves Bessel functions, the terms with $\sigma_a^\pm$ represent analytic continuation for $\mathrm{Im}(W) \leqslant 0$, and

$$H(u_\parallel, u_\perp) = u_\parallel^r \left[\frac{\partial F}{\partial u_\perp} - \frac{K_\parallel}{\gamma W}\left(u_\parallel \frac{\partial F}{\partial u_\perp} - u_\perp \frac{\partial F}{\partial u_\parallel}\right)\right] \times (\gamma W + l + K_\parallel u_\parallel) \qquad (r = 0, 1, \text{or } 2).$$

We should note that the parallel resonance velocities given above may contain a false root because $u^\pm$ are the roots of Eq. (7) rather than $\gamma W - l - K_\parallel u_\parallel = 0$.

The values of $\sigma_a^\pm$, which depend on $u_\perp$, are obtained as follows. For $\mathrm{Im}(W) > 0$, i.e., in the upper half of the complex W-plane, the $u_\parallel$-integration in Eq. (9), $\int_{-\infty}^{\infty} du_\parallel\, H(u_\parallel, u_\perp)/(u_\parallel - u^\pm)$, is to be performed along the real axis, and therefore $\sigma_a^\pm = 0$. For $\mathrm{Im}(W) \leqslant 0$, however, the contour of integration needs to be deformed if $u^\pm$ crosses the real axis of the $u_\parallel$-plane as $\mathrm{Im}(W)$ is varied from a positive to a negative value. Since we are concerned with cases with real $K_\parallel$ and $u_\perp$, the resonance condition $\gamma W - l - K_\parallel u_\parallel = 0$ says that if $u^\pm$ is real then W is also real. It then follows that if W is not real then $u^\pm$ is not real either. Therefore $u^\pm$ does not cross the real axis of the $u_\parallel$-plane unless W crosses the real axis of the W-plane. As W crosses the real axis, we need to determine in which half of the complex $u_\parallel$-plane $u^\pm$ is located. Assuming $|W_i| = |\mathrm{Im}(W)| \ll |Wr| = |\mathrm{Re}(W)|$ and defining

$$D \equiv l^2 + (K_\parallel^2 - W_r^2)(1 + u_\perp^2),$$

we find from Eq. (8),

$$\mathrm{Im}(u_*^\pm) = \begin{cases} \Gamma^\pm W_i, & \text{for } D > 0 \\ \dfrac{W_r}{W_r^2 - K_\parallel^2}\left[\pm(-D)^{1/2} - \dfrac{2lK_\parallel}{W_r^2 - K_\parallel^2} W_i\right], & \text{for } D < 0, \end{cases}$$

where

$$\Gamma^\pm = \frac{1}{W_r^2 - K_\parallel^2}\left\{\pm D^{-1/2}\left[l^2\left(\frac{W_r^2 + K_\parallel^2}{W_r^2 - K_\parallel^2}\right) - K_\parallel^2(1 + u_\perp^2)\right] - \frac{2lK_\parallel W_r}{W_r^2 - K_\parallel^2}\right\}.$$

When $D > 0$, the sign of $\Gamma^\pm$, then, determines whether $u_*^\pm$ lies in the upper or lower half-plane for $\mathrm{Im}(W) > 0$. Therefore, if $\mathrm{Im}(u^\pm)\Gamma^\pm > 0$ for $\mathrm{Im}(W) < 0$, then $u^\pm$ is located in the same half-plane as $u_*^\pm$ is in, and if $\mathrm{Im}(u^\pm)\Gamma^\pm < 0$ for $\mathrm{Im}(W) < 0$ then $u^\pm$ is located in the other half-plane than $u_*^\pm$ is. For $\mathrm{Im}(W) = 0$, we have $\mathrm{Im}(u^\pm) = 0$, i.e., $u^\pm$ is on the real axis.

So, we obtain the following rule:

(a) For $\mathrm{Im}(W)<0$,
  if $\mathrm{Im}(u^{\pm})\Gamma^{\pm}>0$, then $\sigma_a^{\pm}=0$
  if $\mathrm{Im}(u^{\pm})\Gamma^{\pm}<0$, then $\sigma_a^{\pm}=2\,\mathrm{sign}(\Gamma^{\pm})$
(b) For $\mathrm{Im}(W)=0$, $\sigma_a^{\pm}=1\,\mathrm{sign}(\Gamma^{\pm})$

When $D<0$, there occurs no crossing of the real axis as W crosses the real axis. So, we have $\sigma_a^{\pm}=0$ in this case.

Here, we have chosen the parallel and perpendicular momentum $(u_{||}, u_{\perp})$ as the independent variables. One can, of course, choose a different set of variables such as $(\gamma, u_{||})$ or $(\varepsilon, u_{||})$, where ε is energy. We chose $u_{\perp}$ as one of the variables because it enters directly into the argument of Bessel functions (except for a multiplication by $K_{\perp}$) and therefore the values of Bessel functions can be stored for a set of $u_{\perp}$, independent of the other variable. It seems then natural to choose $u_{||}$ as the other variable because it is physically intuitive when approaching the non-relativistic limit.

In performing numerical integrals expressed in Eq. (9) one can immediately see the difficulty associated with the resonance behavior of the integrand with respect to $u_{||}$. To avoid inaccuracy we subtract and add a term proportional to $H(u^{\pm}, u_{\perp})$ as follows:

$$\int du_{||}\frac{H(u_{||},u_{\perp})}{u_{||}-u^{\pm}}=\int du_{||}\frac{H(u_{||},u_{\perp})-H(u^{\pm},u_{\perp})}{u_{||}-u^{\pm}}+H(u^{\pm},u_{\perp})\int\frac{du_{||}}{u_{||}-u^{\pm}}. \tag{10}$$

The integrand of the first term on the right-hand side then varies slowly and smoothly as $u_{||}\to u^{\pm}$, and the integral in the second term can be evaluated as

$$\int_{-\infty}^{\infty}\frac{du_{||}}{u_{||}-u^{\pm}}=\begin{cases} i\pi, & \mathrm{Im}(u^{\pm})>0\\ 0, & =0\\ -i\pi, & <0.\end{cases}$$

Equation (9) can now be written as

$$S=\frac{1}{W^2-K_{||}^2}\int_0^{\infty}du_{\perp}\frac{\mathscr{X}(u_{\perp})}{u^{+}-u^{-}}\left\{\int_{-\infty}^{\infty}du_{||}\times\left[\frac{H(u_{||},u_{\perp})-H(u^{+},u_{\perp})}{u_{||}-u^{+}}-\frac{H(u_{||},u_{\perp})-H(u^{-},u_{\perp})}{u_{||}-u^{-}}\right]+i\pi[(\sigma_r^{+}+\sigma_a^{+})H(u^{+},u_{\perp})-(\sigma_r^{-}+\sigma_a^{-})H(u^{-},u_{\perp})]\right\}, \tag{11}$$

where

$$\sigma_r^{\pm}=\begin{cases}1, & \mathrm{Im}(u^{\pm})>0\\ 0, & =0\\ -1, & <0.\end{cases}$$

The $u_{||}$-integrals in Eq. (11) now seem straightforward. However, care must be taken in evaluating them because the integrand can be a slowly-decaying function as $u_{||}\to\pm\infty$. This is brought about by the subtraction of $H(u^{\pm}, u_{\perp})$ in the numerator. Therefore the integration limit of $u_{||}$ often needs to go much beyond the region where the distribution function has a significant value. To deal with this we divide the $u_{||}$-integral as

$$\int_{-\infty}^{\infty}du_{||}\frac{H(u_{||},u_{\perp})-H(u^{\pm},u_{\perp})}{u_{||}-u^{\pm}}=\left(\int_{-\infty}^{-u_m}+\int_{-u_m}^{u_m}+\int_{u_m}^{\infty}\right)du_{||}\frac{H(u_{||},u_{\perp})-H(u^{\pm},u_{\perp})}{u_{||}-u^{\pm}},$$

where u_m sets the size of the momentum space beyond which there is no significant amount of particles. The first and the last part of the right-hand side can be combined as follows:

$$\left(\int_{-\infty}^{-u_m}+\int_{u_m}^{\infty}\right)du_{||}\frac{H(u_{||},u_{\perp})-H(u^{\pm},u_{\perp})}{u_{||}-u^{\pm}}\approx\left(\int_{-\infty}^{-u_m}+\int_{u_m}^{\infty}\right)du_{||}\frac{-H(u^{\pm},u_{\perp})}{u_{||}-u^{\pm}}=-H(u^{\pm},u_{\perp})\ln\left(\frac{u_m+u^{\pm}}{u_m-u^{\pm}}\right).$$

So, we obtain

$$\int_{-\infty}^{\infty}du_{||}\frac{H(u_{||},u_{\perp})-H(u^{\pm},u_{\perp})}{u_{||}-u^{\pm}}\approx\int_{-u_m}^{u_m}du_{||}\frac{H(u_{||},u_{\perp})-H(u^{\pm},u_{\perp})}{u_{||}-u^{\pm}}-H(u^{\pm},u_{\perp})\ln\left(\frac{u_m+u^{\pm}}{u_m-u^{\pm}}\right). \tag{12}$$

Equation (11) together with Eq. (12), can now be evaluated accurately. Note that the ln (logarithmic) function is a complex function. As long as the integrand varies smoothly and slowly as a function of $u_{||}$ and $u_{\perp}$, we can use an efficient method such as the Gauss quadrature method.

Employing the Gauss quadrature method in both directions, we can write a highly vectorized code. It takes about 10 ms on the Cray-2 to evaluate the conductivity tensor for each cyclotron harmonic with 128×64 quadrature points.

In a non-relativistic case the evaluation of the integral is simpler because there is only one parallel resonant velocity and it is independent of $u_\perp$. In this case the two-dimensional integral takes the form

$$S_{NR} = \int_{-\infty}^{\infty} du_\perp \mathscr{L}(u_\perp) \left\{ \int_{-\infty}^{\infty} du_{||} \frac{G(u_{||}, u_\perp)}{u_{||} - u^0} + i\pi \sigma_a G(u^0, u_\perp) \right\}, \tag{13}$$

where $u^0 = (W - l)/K_{||}$ and

$$G(u_{||}, u_\perp) = u_{||}^r \left[\frac{\partial F}{\partial u_\perp} - \frac{K_{||}}{W} \left(u_{||} \frac{\partial F}{\partial u_\perp} - u_\perp \frac{\partial F}{\partial u_{||}} \right) \right]$$

($r = 0$, 1, or 2). This integral can be evaluated in exactly the same manner as in relativistic cases. Furthermore, the value of σ_a is obtained in the same manner as in relativistic cases: $\sigma_a = 2\ \text{sign}(K_{||})$ for $\text{Im}(W) < 0$, and $\sigma_a = 1\ \text{sign}(K_{||})$ for $\text{Im}(W) = 0$.

To solve the dispersion relation, Eq. (1), we employ a NAG library routine, C05NBF, which obtains the Jacobian numerically [13]. Given a good initial guess, the code finds a root in about 10 iterations.

3. SPLINE FIT OF A DISTRIBUTION

The numerical algorithm described in the previous section should work well as long as the distribution is smooth and there are an adequate number of quadrature points. When a distribution function cannot be easily approximated by a combination of analytic functions as described earlier, we employ a spline fit to the function given on a numerical grid, e.g., the $(u_{||}, u_\perp)$ rectangular grid. We represent the distribution by cubic splines as

$$F(x, y) = \sum_{i=1}^{4} \sum_{j=1}^{4} c_{ij} \left(\frac{x - x_I}{x_{I+1} - x_I} \right)^{i-1} \times \left(\frac{y - y_J}{y_{J+1} - y_J} \right)^{j-1} \tag{14}$$

for a grid cell defined by $x_I \leqslant x \leqslant x_{I+1}$ and $y_J \leqslant y \leqslant y_{J+1}$, where x and y correspond to $u_{||}$ and $u_\perp$, respectively, and the c's are spline coefficients.

To approximate the analytic continuation of F into the complex $u_{||}$ plane we replace x in Eq. (14) by a complex value.

If the data values for the distribution are noisy, we first smooth them by a digital filtering technique [14]. We have found a binomial filter given by

$$F_{I,J}^{\text{new}} = \tfrac{1}{16}[F_{I-1,J-1} + 2F_{I,J-1} + F_{I+1,J-1} + 2F_{I-1,J} + 4F_{I,J} + 2F_{I+1,J} + F_{I-1,J+1} + 2F_{I,J+1} + F_{I+1,J+1}]$$

quite satisfactory.

4. EXAMPLE RESULTS

4.1. *Cyclotron–Maser Instability*

Cyclotron–maser instability is an electromagnetic instability with an X-mode polarization and predominantly perpendicular propagation to the external magnetic field [15]. This instability arises only in a relativistic treatment and requires a loss cone distribution.

We employ the DGH-type distribution function,

$$F(u_{||}, u_\perp) = \frac{1}{\pi^{3/2}\alpha^3 m!} \left(\frac{u_\perp}{\alpha} \right)^{2m} \exp\left[-\frac{u_\perp^2 + u_{||}^2}{\alpha^2} \right], \tag{15}$$

to test the code against a published result. In Fig. 1 we present a comparison with the result of Pritchett [16] for

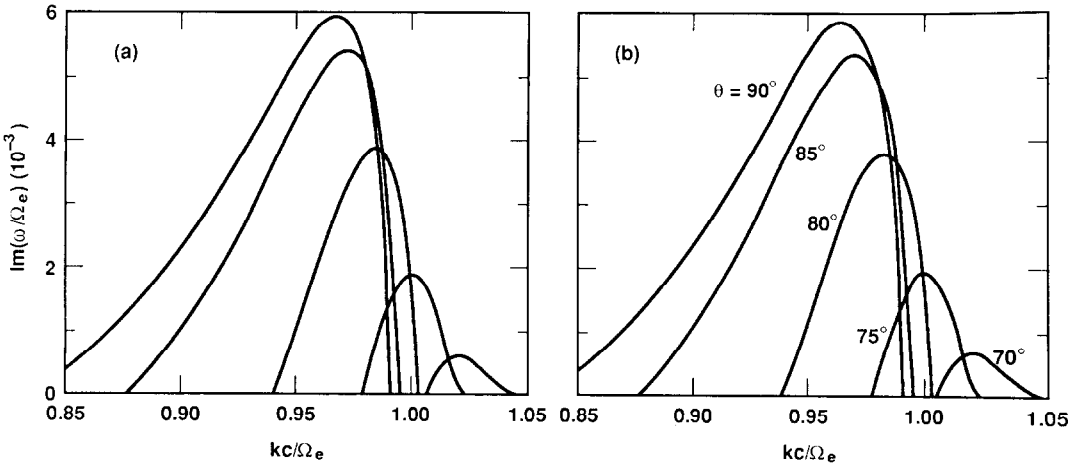

FIG. 1. (a) Growth rate vs. wavenumber for cyclotron–maser instability from the code. (b) The results of Pritchett (Fig. 8 of Ref. [16]).

the growth rate vs the wave number. The parameters chosen are: $m = 2$, $\alpha = 0.1$, $\omega_{pe}/\Omega_e = 0.05$, and propagation angle $\theta = 70°$–$90°$. In this comparison we have used the analytical derivatives of the function given by Eq. (15) to calculate the integrand on the quadrature points.

We find a very good agreement between the two results. In terms of numerical values, the growth rate, ω_i/Ω_e from our code is 5.93×10^{-3} compared to 5.94×10^{-3} and the frequency, ω_r/Ω_e, is 0.9956 compared to 0.9958 at the wavenumber $ck/\Omega_e = 0.966$ and $\theta = 90°$. In this case it took about 0.3 s to find a root.

In this example, the characteristic velocity of electrons, v, is much smaller than the velocity of light, c. Therefore, it may not seem to provide a valid justification of our method in relativistic cases. However, the condition, $v/c \ll 1$, is not a sufficient condition for the validity of a non-relativistic calculation, as demonstrated in Ref. [15], because $(v/c)^2$ must be compared to the fractional shift of the wave frequency away from cyclotron resonance to determine whether the relativistic mass shift must be included in the resonant denominator. Cyclotron–maser instability has very small shifts and requires a relativistic calculation even at very low temperatures. In fact, cyclotron–maser instability does not occur at any temperature in a non-relativistic theory. In addition, there are no restrictions or assumptions implied in our method regarding the value of v/c.

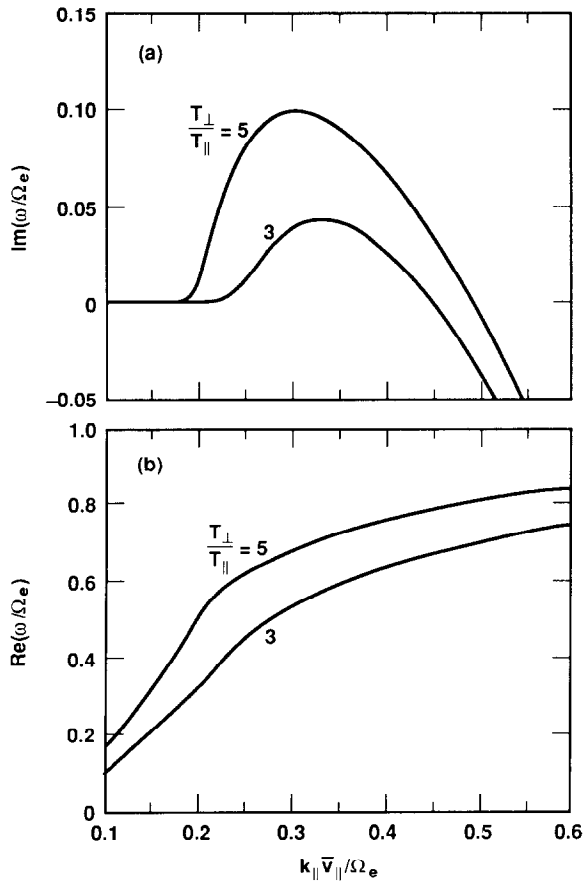

FIG. 2. (a) Growth rate and (b) frequency vs. wavenumber for whistler instability from the code. Parameters are: $\omega_{pe}/\Omega_e = 8$, $\beta_\perp = 0.3$, and $T_\perp/T_\parallel = 5$ and 3. $\bar{v}_\parallel$ $[= (2T_\parallel/m)^{1/2}]$ is the thermal velocity in the parallel direction.

TABLE I

Comparison of Growth Rates from Two Calculations: One by Use of a Numerical Distribution and the Other by Use of Eq. (4)

$ck_\parallel/\Omega_e$	Im ω/Ω_e (numerical)	Im ω/Ω_e (Use of Eq. (4))
5.5	6.800×10^{-4}	6.803×10^{-4}
7.0	2.909×10^{-2}	2.908×10^{-2}
8.5	4.300×10^{-2}	4.299×10^{-2}
10	2.828×10^{-2}	2.828×10^{-2}

Note. Parameters are $\omega_{pe}/\Omega_e = 8$, $\beta_\perp = 0.3$, and $T_\perp/T_\parallel = 3$.

4.2. *Whistler Instability*

We chose whistler instability as another example to test the code. The whistler-wave instability was shown to arise when the electron perpendicular temperature is larger than the parallel temperature [17].

In this case we use a bi-Maxwellian distribution given by Eq. (3). The derivatives of the distribution function were first evaluated on a rectangular grid in velocity space and a cubic spline fit was obtained for each derivative as described in Section 3. The integrands were then evaluated on the quadrature points from the fit given by Eq. (14).

In this example we have treated a non-relativistic case to compare the results with the roots obtained from the dispersion relation employing Eq. (4) which is expressed in terms of the plasma dispersion function and the modified Bessel functions.

The frequency and growth rate are shown in Fig. 2. These were obtained by our method along with the use of the spline fit to the distribution. The parameters chosen are: $\omega_{pe}/\Omega_e = 8$, $\beta_\perp = 8\pi n T_\perp/B^2 = 0.3$ and $T_\perp/T_\parallel = 3$ and 5. These values are those used by Gladd [18]. Our results are somewhat different from Gladd's result (Fig. 2 in Ref. [18]) due to his approximation, $ck/\omega \gg 1$. For comparison, Table I lists the growth rates shown in Fig. 2 and those obtained from the use of the analytical expression, Eq. (4), in the dispersion relation. We find very good agreement between the two results, which demonstrates the validity of our method.

5. CONCLUSION

We have presented a new numerical method for studying a microinstability of a uniform plasma in a magnetic field.

Our method is a very general one that can be applied to either a relativistic or non-relativistic plasma with a completely arbitrary distribution function. By presenting the results for cyclotron–maser instability and whistler instability we have shown that our method is a valid and efficient one; we have shown that the results from our code agree very well with the previously published results or analytical calculations.

A method based on the cyclotron harmonic expansion, such as ours, may seem less efficient than Weiss' method. However, use of a multi-processor machine alleviates the inefficiency because each harmonic term can be evaluated independently on each processor. The method should prove useful in many applications because one can use any distribution including those defined only on a numerical grid. One such application may be space plasmas for which distribution functions measured in situ are available.

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